THE OCTOBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE two hundred and fifth regular meeting of the Society was held at Columbia University on Saturday, October 25, 1919. The following forty-eight members attended the two sessions:

Dr. J. W. Alexander, Professor R. C. Archibald, Dr. I. A. Barnett, Dr. Charlotte C. Barnum, Professor A. A. Bennett, Professor W. J. Berry, Professor G. D. Birkhoff, Professor Pierre Boutroux, Dr. G. A. Campbell, Professor F. N. Cole, Dr. Tobias Dantzig, Professor L. P. Eisenhart, Professor T. S. Fiske, Professor W. B. Fite, Dr. T. H. Gronwall, Professor C. O. Gunther, Professor Olive C. Hazlett, Dr. A. A. Himwich, Mr. S. A. Joffe, Professor Edward Kasner, Mr. H. P. Kean, Professor O. D. Kellogg, Professor C. J. Keyser, Dr. K. W. Lamson, Professor P. H. Linehan, Professor James Maclay, Professor R. L. Moore, Professor G. W. Mullins, Mr. George Paaswell, Dr. Alexander Pell, Professor Anna J. Pell, Dr. G. A. Pfeiffer, Professor H. W. Reddick, Dr. J. F. Ritt, Dr. Caroline E. Seely, Professor D. E. Smith, Professor P. F. Smith, Dr. J. M. Stetson, Mr. J. J. Tanzola, Professor H. D. Thompson, Professor Oswald Veblen, Mr. A. C. Washburne, Mr. H. E. Webb, Professor J. H. M. Wedderburn, Professor Mary E. Wells, Mr. R. A. Wetzel, Mr. J. K. Whittemore, Professor A. H. Wilson.

Vice-President G. D. Birkhoff occupied the chair. The Council announced the election of the following persons to membership in the Society: Dr. C. C. Camp, University of Illinois; Professor C. J. Coe, University of Michigan; Dr. Teresa Cohen, Johns Hopkins University; Mr. W. E. Heal, U. S. Bureau of Plant Industry; Dr. C. A. Nelson, University of Kansas; Mr. J. L. Walsh, Harvard University. Four applications for membership in the Society were received.

A committee was appointed to audit the accounts of the Treasurer for the current year. A list of nominations for officers and other members of the Council was adopted and ordered printed on the official ballot for the annual election. Propositions for establishing a board of custodians of the property of the Society and for issuing a new catalogue of the library were laid over for action at the annual meeting.

The following papers were read at the October meeting:

(1) Mr. J. K. WHITTEMORE: "Motion in a resisting medium."

(2) Miss ANNA M. MULLIKIN: "A countable collection of mutually exclusive closed point sets with connected sum."

(3) Professor OLIVE C. HAZLETT: "New proofs of certain finiteness theorems in the theory of modular covariants."

(4) Professor R. D. CARMICHAEL: "On the convergence of certain classes of series of functions."

(5) Professor R. D. CARMICHAEL: "Note on a transformation of series similar to the principle of inversion in the theory of numbers."

(6) Professor R. D. CARMICHAEL: "Note on the theory of integral functions of the first class."

(7) Professor ANNA J. PELL: "Two linear integral equations with two parameters."

(8) Professor G. D. BIRKHOFF: "Note on stable periodic orbits."

(9) Professor O. D. KELLOGG: "Some problems connected with submarine acoustics."

(10) Dr. T. H. GRONWALL: "Differential variations in ballistics, with applications to the qualitative properties of the trajectory."

(11) Professor A. A. BENNETT: "Standard density, temperature and pressure of air aloft."

(12) Professor A. A. BENNETT: "The probable error of a small number of rounds."

(13) Professor A. A. BENNETT: "The physical bases of ballistic table computations."

(14) Professor A. A. BENNETT: "The sign of the distance in analytical geometry."

(15) Dr. E. F. SIMONDS: "Invariants of infinite groups in the plane."

(16) Professor EDWARD KASNER: "The motion of n bodies, under any forces, starting from rest."

The paper of Miss Mullikin was communicated to the Society and read by Professor R. L. Moore. Dr. Simonds's paper was read by Professor Kasner. The papers of Professor Carmichael and the last three papers of Professor Bennett were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Mr. Whittemore's paper is devoted to the study of the motion of a particle whose tangential acceleration is a given function of the velocity alone, $a = F(v)$. It is proved that if λ is a simple positive root of $F(v) = 0$ and $F(v)$ is subject to certain other conditions, generally satisfied in a real problem, $\lambda - v$ approaches zero at least as rapidly as e^{-mt}, where t is the time and m a positive constant; that under the same conditions $\lambda t - x$, where x is the distance moved, approaches a limit L, and differs from L by a quantity not greater than a constant multiple of e^{-mt}. When the root λ is not simple the results are different. Various examples are considered and some numerical illustrations given. Four problems are discussed in which there act forces working at a constant rate. Such problems seem to have been neglected in books on elementary mechanics, but have nevertheless some interest. Especial study is made of the starting of a steamer driven by engines developing constant horsepower.

2. The sum of a finite number (more than one) of mutually exclusive closed point sets is never connected.* Nor is the sum of a countably infinite number of such point sets connected if each of them consists of only one point. In one dimension no countably infinite collection of mutually exclusive closed point sets ever has a connected sum. One might rather naturally be inclined to believe that this proposition holds true also in two dimensions. Miss Mullikin shows by an example that this is, however, not the case.

3. Professor Hazlett's paper gives new proofs of certain finiteness theorems in the theory of modular invariants. In 1909 Professor Dickson proved the finiteness theorem for modular invariants in the Galois field $GF[p^n]$. This was followed in 1913 by his proof of the finiteness theorem for modular covariants. The next year, Professor Wiley proved the finiteness theorem for modular invariants of a system of forms over the Galois field $GF[p]$ and any number of cogredient points (x_i, y_i). This proof used a lemma which is closely related to Hilbert's well known theorem on a set of polynomials.

* A set of points is said to be connected if, however it be divided into two mutually exclusive subsets, one of them contains a limit point of the other one.

In a paper presented at Chicago in December, 1918, the author proved a theorem which enables us to construct all modular covariants of a system S of forms from the modular invariants of an enlarged system S' consisting of the forms of S and an additional linear form. By using this theorem, and the finiteness theorem for modular invariants, we can prove at once the finiteness theorem for modular covariants as well as the more general theorem proved by Professor Wiley. The present proof gives a method for obtaining from a fundamental set of invariants of S' a set of modular covariants of S in terms of which all modular covariants of S are expressible as polynomials.

Finally, by means of a symbolic notation, the author proves from this theorem that every formal covariant of a system of forms over the Galois field $GF[p^n]$ is a polynomial in the modular covariants which have been made formally invariant as to the coefficients of the forms and in the irreducible covariants which are congruent to zero whenever the coefficients are marks of the field. This theorem is verified for the binary quadratic modulo 3 and for the binary cubic modulo 2.

4. In this paper Professor Carmichael develops the first fundamental convergence properties of certain very general classes of series including many of those which are classic in analysis, such as the power series, factorial series, and Dirichlet series.

5. A certain general principle of transformation of series analogous to the principle of inversion in the theory of numbers is here employed by Professor Carmichael to derive a few classic results; and the principle itself is extended and more general results are obtained through use of generalizations of the λ-function often employed in the theory of the distribution of prime numbers.

6. The object of this note by Professor Carmichael is to extend certain of the classic theorems about integral functions. It contains a modification of the Weierstrass factor theorem valid in certain cases and some consequences relative to the relation between the zeros of an integral function and those of its derivative.

7. It is shown in Mrs. Pell's paper that there exist real values of λ and μ for which the two linear integral equations

$$u(x) = \lambda \int_a^b K(x, y)u(y)dy + \mu \int_a^b L(x, y)u(y)dy,$$

$$v(s) = \lambda \int_c^d M(s, t)v(t)dt - \mu \int_c^d N(s, t)v(t)dt$$

have solutions $u(x) \not\equiv 0$ and $v(s) \not\equiv 0$, if K, L, M, N are continuous and symmetric and L, N are positive definite. Functions of two variables, which can be expressed in a certain form, can be expanded into a uniformly convergent series in terms of the functions of a system adjoint to the system $\{u_i(x)v_i(s)\}$.

8. The object of this note by Professor Birkhoff is to establish that infinitely many periodic orbits of certain types exist in the infinitesimal vicinity of any stable periodic orbit.

The note will appear in the *Proceedings of the National Academy of Sciences*.

9. Professor Kellogg outlined a number of mathematical problems which have arisen in connection with the detection of submerged submarines by sound.

10. Dr. Gronwall's paper gives a method for computing differential variations, based on the discovery of a new algebraic integral of Bliss's adjoint system of differential equations. By means of this integral, the third order system of Bliss reduces to a linear differential equation of the second order, and the numerical computation of the differential variations is materially shortened both for ordinary and anti-aircraft trajectories.

The linear differential equation referred to being of the second order, the general behavior of its solutions can be determined without much difficulty, and there result a large number of qualitative properties of the trajectory, for instance the following: the lateral deflection of the projectile due to a constant cross wind is always less than the increase in range due to a following wind of the same velocity.

11. This paper by Professor Bennett discusses the special question of the choice of certain functions assumed to represent standard density, temperature and pressure of air aloft as functions of the altitude measured in meters. The necessity for such standard functions and the reason for the exponential forms selected as against meteorological means, are explained, as applied to the problem of exterior ballistics.

12. Many known variables are taken explicitly into account to secure as great accuracy as possible in firing heavy artillery. There remains, however, an inevitable dispersion, upon which also computations are based through the magnitude of the probable errors in range and deflection. As an ordnance problem, the probable error of the various types of ammunition must be measured. A consideration of the accuracy of the probable error when based for the sake of economy on a relatively small number of rounds is here investigated by Professor Bennett, along the lines laid out by Czuber in his standard treatises Beobachtungsfehler and Wahrscheinlichkeitsrechnung. The paper will appear shortly in the *Journal of the U. S. Artillery*.

13. A pamphlet entitled "The Physical Bases of Ballistic Table Computations" is shortly to appear as an Ordnance Pamphlet, in the series "Notes on the Construction of Ordnance." This pamphlet by Professor Bennett involves no mathematical computations on the part of the reader and is rather a critical résumé of the latest practices in ballistic theory in so far as they deal with the topic mentioned. It is designed for use as a reference text in courses on gunnery, and will also constitute a part of the general introduction to the ballistic tables now being prepared under the supervision of the author.

14. In this paper Professor Bennett discusses the elementary question of the definition of distance in analytical geometry. The distance between points and between a point and a line are assumed as defined up to the algebraic sign. Three simple conventions are possible, (1) the arithmetical, by means of which all distances are positive, (2) the transcendental, by which distances are measured vectorially with a convention arbitrarily fixed in each case, (3) the algebraic, in which either

sign is always possible. Each definition has its objections, the most common usage being perhaps the least convenient. The difficulties are illustrated by very simple examples. The paper appeared in the October number of the *American Mathematical Monthly*.

15. In his Columbia dissertation, published in the *Transactions* (1918), Dr. Simonds studied the order and the number of invariants of differential configurations; and in particular with respect to the group of all point transformations obtained results entirely different from those given by Rabut in his paper on "Invariants universelles" (1898), but consistent with Kasner's invariant of the second order (*American Journal of Mathematics*, 1906).

In the present paper, Dr. Simonds discusses the five types of infinite groups of point transformations and the three types for contact transformations, according to Lie's classification, finding for each type the smallest number of curves, with distinct tangents, through a common point, which have an invariant of any given order n. For example, under the conformal group, two curves obviously have an invariant of first order, four curves have one of second order (checking with a result given by Kasner), three curves have an invariant for each of the orders 3, 4, 5, $\cdots$. For each type of group the simplest invariants are given explicitly.

16. In his discussion of the general geometric properties of systems of dynamical trajectories, Professor Kasner incidentally obtained the simple theorem that any particle starting from rest in any positional field of force describes a trajectory whose initial curvature is one third the curvature of the line of force passing through the starting point (*Transactions*, 1905, Princeton Colloquium Lectures 1913, page 9). This result holds for space of any dimensionality. In the present paper the theorem is extended to the motion of any number of particles acted upon by any positional forces, conservative or non-conservative. The ratio 1:3 remains valid even in the case of a resisting medium provided the resistance approaches zero when the velocity approaches zero.

F. N. COLE,
Secretary.

THE OCTOBER MEETING OF THE SAN FRANCISCO SECTION.

THE thirty-fourth regular meeting of the San Francisco Section was held at the University of California on Saturday, October 25. There were two sessions. The morning session was opened by the chairman of the Section, Professor Cajori, who later was relieved by Professor Blichfeldt.

The attendance was twenty-one, including the following fourteen members of the Society:

Professors R. E. Allardice, B. A. Bernstein, H. F. Blichfeldt, Thomas Buck, Florian Cajori, M. W. Haskell, L. M. Hoskins, D. N. Lehmer, W. A. Manning, H. C. Moreno, Dr. F. R. Morris, Professors C. A. Noble, T. M. Putnam, and Dr. Pauline Sperry.

The following officers were elected for the year: chairman, Professor H. F. Blichfeldt; secretary, Professor B. A. Bernstein; programme committee, Professors W. A. Manning, D. N. Lehmer, and B. A. Bernstein.

The dates of the next two meetings were fixed as April 10, 1920, and October 23, 1920.

The following papers were presented:

(1) Professor W. A. MANNING: "Doubly transitive groups with transitive subgroups of lower degree."

(2) Professor M. W. HASKELL: "On self-dual curves" (preliminary report).

(3) Professor W. A. MANNING: "Groups of degree n in which there is a circular permutation of n letters."

(4) Professor FLORIAN CAJORI: "Surveying and astronomical instruments used in America before the nineteenth century."

(5) Professor E. T. BELL: "On a certain inversion in the theory of numbers."

(6). Professor E. T. BELL: "On the enumeration of proper and improper representations in homogeneous forms."

(7) Professor E. T. BELL: "On proper and improper representations in certain quadratic forms of Liouville."

(8) Professor R. M. WINGER: "Some generalizations of the satellite theory."

In the absence of the authors the papers of Professors Bell and Winger were read by title. Abstracts of the papers follow below.

1. Professor Manning presented the following theorem:

Let a doubly (but not triply) transitive group of degree n have transitive subgroups $H, H_1, \cdots, H_r$ of degrees $m, m + q_1, \cdots, m + q_1 + \cdots + q_r$, respectively, and of no other degree $< n$ and $> m$. Then H_{i-1} $(i = 1, 2, \cdots, r;\ H_0 = H)$ has at least

$$1 + (q_i + q_{i+1} + \cdots + q_u)\left[\frac{1}{q_{u+1}} + \frac{1}{q_{u+2}} + \cdots + \frac{1}{q_r} + 1\right]$$

systems of imprimitivity of q_u letters which have one letter in common $(u = i, i + 1, \cdots, r)$. When $u = r$, no two of the above systems of q_r letters that have one letter in common have a second letter in common.

In particular this theorem asserts that H has $1 + q_1 + q_2 + \cdots + q_r$ systems of q_r letters with one letter in common.

This problem was originated by C. Jordan. He showed (1871) that H_r, and a fortiori $H, H_1, \cdots$, must have at least two systems of imprimitivity of q_r letters with one letter in common. Then B. Marggraff (1889) proved that H_r has $1 + q_r$ systems of imprimitivity of q_r letters with one letter in common. This result, as Marggraff did not fail to notice, reveals the presence in H_{i-1} of at least $1 + q_u/q_{u+1}$ systems of q_u letters with q_{u+1} letters in common $(u = i, i + 1, \cdots, r - 1)$. It was subsequently shown (Manning, 1906) that systems of imprimitivity of H_{i-1} of q_u letters each, with q_{u+1} letters in common, can be chosen in at least

$$1 + q_u\left[\frac{1}{q_{u+1}} + \frac{1}{q_{u+2}} + \cdots + \frac{1}{q_r} + 1\right]$$

ways $(u = i, i + 1, \cdots, r)$. The corresponding result, as stated in the present theorem, was suggested by a study of the group of isomorphisms of the elementary group of order 2^n.

2. Professor Haskell gave a preliminary account of the self-dual curves which are self-reciprocal with respect to a conic. He showed that any conic is self-reciprocal with respect to

each of a doubly infinite set of conics, and that this relation is mutual; that the self-dual cubic is self-reciprocal with respect to a singly infinite set of conics; and that the quintic with five cusps is self-reciprocal with respect to a single conic.

3. It was proved by Burnside that a simply transitive group of degree p (p a prime) is cyclic or contains an invariant subgroup of order p. He also proved that a simply transitive group of degree p^n in which there is a permutation of order p^n is necessarily imprimitive and compound. In his second paper Professor Manning announced the more general theorem:

If a simply transitive group of composite order contains a circular permutation of all its letters, it is compound.

4. Professor Cajori pointed out that the surveying and astronomical instruments used in America before 1800 were almost altogether from English makers—Heath, Short, Nairne, Bird, Sisson, Dollond, Ramsden. The chief American instrument makers were David Rittenhouse and Thomas Godfrey. Photographs of instruments were exhibited.

5. The inversion considered in Professor Bell's first paper is as follows. Define $f(x)$ to be regular if it exists and has a finite value when x is an integer > 0, and $f(1) \neq 0$. Then there exists a unique regular f', called the inverse of f, such that

$$\Sigma f(d)f'(n/d) = f(1) \text{ or } 0$$

according as $n = 1$ or $n > 1$, the Σ extending to all divisors d of the arbitrary integer $n > 0$. The f' thus defined is distinct from Berger's inverses. The theorem, which in some respects is similar to the well known inversion of H. F. Baker, is useful in many parts of arithmetic. The paper will appear in the *Tôhoku Mathematical Journal*.

6. Algebraic processes, elliptic function or other, do not easily yield the number of proper representations of an integer in a homogeneous form, but when applicable at all, give the total number of representations with great readiness. In the arithmetical treatment, it has been customary to deduce the total number of representations from the number of

proper representations. Inverting this procedure, Professor Bell, in his second paper, proves formulas of remarkable simplicity for the expression of the proper number in terms of the total, so that if the latter is within reach of analysis, so also now is the former. A few illustrations are given in the derivation of new results concerning 6, 8, 10 or 12 squares and other simple quadratic forms. The cases of 10, 12 squares present some unexpected singularities.

7. From 1860 to 1864 Liouville published in his *Journal* numerous theorems on the number of total and proper representations of integers in special quadratic forms of four and six indeterminates. His formulas for total numbers of representations were for the most part proved in 1890 by Pepin, those omitted being readily demonstrable by elliptic functions and other algebraic means. The formulas for proper representations have not hitherto been proved. Modifying the general principles of his second paper to fit Liouville's forms, Professor Bell demonstrates all of the unproved results very simply. The paper will appear in the *Journal de Mathématiques pures et appliquées*, (in French), and a fuller abstract shortly in the Paris *Comptes Rendus*.

8. Professor Winger's paper appeared in full in the November BULLETIN.

B. A. BERNSTEIN,
Secretary of the Section.

ON THE PROOF OF CAUCHY'S INTEGRAL FORMULA BY MEANS OF GREEN'S FORMULA.

BY MR. J. L. WALSH.

(Read before the American Mathematical Society December 30, 1919.)

IT is well known that Cauchy's integral formula for an analytic function $f(z) = u(x, y) + iv(x, y)$ of the complex variable $z = x + iy$ is analogous to Green's formula for the functions u and v, and that moreover Cauchy's formula can be proved from Green's formula. Picard (Traité d'Analyse (1905), volume II, page 114) gives this proof *assuming that*

u and v have continuous second partial derivatives, which fact is proved later by means of Cauchy's formula. It is the purpose of the present note to give a modification of Picard's proof which shall not be open to this objection.*

Two functions $U(x, y)$ and $U'(x, y)$ are said to be conjugate in a region T of the x, y-plane if within that region they are continuous with their first partial derivatives and if in addition

$$\frac{\partial U}{\partial x} = \frac{\partial U'}{\partial y}, \qquad \frac{\partial U}{\partial y} = -\frac{\partial U'}{\partial x}. \tag{1}$$

A function $V(x, y)$ is said to be harmonic in T if in T it is continuous together with its first and second partial derivatives and if it satisfies Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0.$$

Suppose U and U' are conjugate in a region T, and that V and V' are not only conjugate but also harmonic in T. Then if C is any contour bounding a region wholly in T, we shall have

$$\int_c \left[U\frac{\partial V}{\partial n} - V\frac{\partial U}{\partial n} \right] ds = 0. \tag{2}$$

For the analysis given by Picard (l. c., page 10) establishes the following relations, where the double integrals are extended over the interior of C:

$$\begin{aligned} \iint \left[\frac{\partial U}{\partial x}\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}\frac{\partial V}{\partial y} \right] dxdy &= -\int_c U\frac{\partial V}{\partial n} ds, \\ \iint \left[\frac{\partial U'}{\partial x}\frac{\partial V'}{\partial x} + \frac{\partial U'}{\partial y}\frac{\partial V'}{\partial y} \right] dxdy &= -\int_c U'\frac{\partial V'}{\partial n} ds. \end{aligned} \tag{3}$$

* Osgood, Funktionentheorie (1912), p. 679, gives such a proof due to Morera, but requiring the existence of a Green's function for the region. He also points out that Green's formula can be derived from that of Cauchy

A proof for the circle, and hence for any region which can be conformally represented on the circle, is given by Kellogg, this BULLETIN, vol. 10 (1903-04), p. 255.

Compare also J. L. Walsh, "Note on Cauchy's integral formula," *Annals of Mathematics*, vol. 18, p. 79, where Cauchy's formula is proved by means of Cauchy's integral theorem and a proof of the mean value theorem for harmonic (or conjugate) functions due to Bôcher.

But by virtue of (1) and the corresponding relations between V and V', the left-hand members of the two equations (3) are identical. Hence

$$\int_c U\frac{\partial V}{\partial n}ds = \int_c U'\frac{\partial V'}{\partial n}ds.$$

Also

$$0 = \int_c \frac{\partial(U'V)}{\partial s}ds = \int_c \left[U'\frac{\partial V}{\partial s} + V\frac{\partial U'}{\partial s}\right]ds$$

$$= \int_c U'\frac{\partial V'}{\partial n}ds - \int_c V\frac{\partial U}{\partial n}ds.$$

This immediately gives us equation (2).

Having proved this theorem,* we can continue by means of the analysis of Picard (l. c., page 15), setting $V = \log r$ and proving

$$U(a, b) = \frac{1}{2\pi}\int_c \left[\log r\frac{\partial U}{\partial n} - U\frac{\partial \log r}{\partial n}\right]ds.$$

Further developments (l. c., page 114) give us easily Cauchy's integral formula

$$f(z') = \frac{1}{2\pi i}\int_c \frac{f(z)}{z - z'}dz.$$

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* This was proved for a region bounded by a single contour, but there is no real difficulty involved in its later application to the multiply connected region.

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A SET OF COMPLETELY INDEPENDENT POSTULATES FOR THE LINEAR ORDER η*.

BY PROFESSOR M. G. GABA.

(Read before the American Mathematical Society September 4, 1919.)

PROFESSOR E. V. HUNTINGTON has published† three sets of completely independent postulates for serial order. His set A involves four postulates, which is as high a number of postulates as had been proved completely independent. In the present paper are given seven postulates which form a categorical and completely independent set for the linear order.

Our basis is a class of elements $[p]$ and an undefined dyadic relation (called 'less than') among the elements. If we are given two elements p_1p_2 and if the relation p_1 less than p_2 holds, we will symbolize it by $p_1 < p_2$. If the relation p_1 less than p_2 does not hold, we will symbolize it by $p_1 \not< p_2$.

Our postulates are:

I. If $p_1 < p_2$, then $p_2 \not< p_1$.

II. If $p_1 \not< p_2$, then $p_2 < p_1$; p_1, p_2 distinct.

III. If $p_1 < p_2$ and $p_2 < p_3$, then $p_1 < p_3$.

IV. If $p_1 < p_2$, then there exists a p_3 such that $p_1 < p_3$ and $p_3 < p_2$.

V. For every p_1 there exists a p_2 such that $p_2 < p_1$.

VI. For every p_1 there exists a p_2 such that $p_1 < p_2$.

VII. The class of elements $[p]$ form a denumerable set.

That the set is categorical follows from the fact that the seven postulates stated are the necessary and sufficient conditions for the linear order η. To show complete independence it will be necessary to cite 128 (2^7) examples showing all possible combinations ($\pm\pm\pm\pm\pm\pm\pm$) of our postulates holding and not holding. This is done by giving eight definitions of $<$, and sixteen sets of points such that each definition is applicable to every one of the sets, and every combination

* The linear order η is an ordered set equivalent to that of all the rational numbers.

† "Sets of completely independent postulates for serial order." This BULLETIN, March, 1917. This paper contains a bibliography of complete independence.

of definition of $<$ and set yields a different example. The eight definitions give the eight $(\pm\pm\pm)$ groups of cases for the implicational postulates I, II and III, whereas each of the sixteen sets gives all the eight cases where any particular set $(\pm\pm\pm\pm)$ of the existential postulates IV, V, VI and VII hold or do not hold.

For the independence examples, the set $[p]$ consists of points on a line such that

	IV	V	VI	VII			
1)	−	−	−	−	$p = -3,$	$-2 \leqq p \leqq 2,\ p = 3$	and p real.
2)	−	−	−	+	$p = -3,$	$-2 \leqq p \leqq 2,\ p = 3$	and p rational.
3)	−	−	+	−	$p = -3,$	$-2 \leqq p < 3,$	and p real.
4)	−	−	+	+	$p = -3,$	$-2 \leqq p < 3,$	and p rational.
5)	−	+	−	−		$-3 < p \leqq 2,\ p = 3$	and p real.
6)	−	+	−	+		$-3 < p \leqq 2,\ p = 3$	and p rational.
7)	−	+	+	−	$-3 < p \leqq 1/2,$	$2 \leqq p < 3,$	and p real.
8)	−	+	+	+	$-3 < p \leqq 1/2,$	$2 \leqq p < 3,$	and p rational.
9)	+	−	−	−		$-3 \leqq p \leqq 3,$	and p real.
10)	+	−	−	+		$-3 \leqq p \leqq 3,$	and p rational.
11)	+	−	+	−		$-3 \leqq p < 3,$	and p real.
12)	+	−	+	+		$-3 \leqq p < 3,$	and p rational.
13)	+	+	−	−		$-3 < p \leqq 3,$	and p real.
14)	+	+	−	+		$-3 < p \leqq 3,$	and p rational.
15)	+	+	+	−		$-3 < p < 3,$	and p real.
16)	+	+	+	+		$-3 < p < 3,$	and p rational.

A definition of $<$ requires that whenever we are given two numbers of our set p_1p_2 we have a criterion whereby we can tell whether the relation $p_1 < p_2$ holds or does not hold. In all the eight definitions of $<$ the relation holds for any pair of numbers p_1p_2 if it holds in the case of ordinary linear order,

	I	II	III	
1′)	−	−	−	except $0 \not< 1,\ -1 < -2,\ 0 < -1$ and $0 < -2$.
2′)	−	−	+	except $1 < -1,\ 1 < 0,\ 0 < -1,\ p_1 < -1,\ p_1 < 0,\ p_1 < 1,\ -1 \not< p_2,\ 0 \not< p_2$ and $1 \not< p_2;\ p_1 \neq -1, 0, 1;\ p_2 \neq -1, 0, 1.$
3′)	−	+	−	except $0 < -m/2^n$, n positive integer and m odd positive integer.
4′)	−	+	+	except $p_1 < -1,\ p_1 < 0,\ p_1 < 1,\ -1 \not< p_2,\ 0 \not< p_2$, and $1 \not< p_2;\ p_1 \neq 3;\ p_2 \neq -1, 0, 1.$
5′)	+	−	−	and $p_2 - p_1 < 1/3.$
6′)	+	−	+	and $p_2 - p_1 = m/2^n$, n positive integer and m odd integer.
7′)	+	+	−	except $0 < -m/2^n$ and $-m/2^n \not< 0$, n positive integer and m odd positive integer.
8′)	+	+	+	with no exceptions.

To illustrate: The independence example where postulates II, III, V, and VII hold and postulates I, IV and VI do not hold $(-++-+-+)$ is definition 4′ used on set 6.

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CERTAIN PROPERTIES OF BINOMIAL COEFFICIENTS.

BY PROFESSOR W. D. CAIRNS.

(Read before the American Mathematical Society September 4, 1919.)

KENYON* has generalized results given by Chrystal† and others for the sum of products of equal powers of the terms of an arithmetic progression and the coefficients of $(x-y)^n$. In evaluating certain integrals connected with the probability curve,‡ the author has had to sum a similar set of products for the coefficients of $(x+y)^{2n}$; each binomial coefficient is multiplied by a given power of the number of that term from the middle term, e.g., the $(n+1)$th term by 0^q, the nth term by $(-1)^q$, etc., and the results added. One feature of this paper is the distinct gain from the symmetry thus utilized. A recursion formula is obtained for the sums and an expression for the term of highest degree in n, which is all that is needed in the desired application. For odd values of q only the terms from the middle to the end are used.

§1.

The general binomial coefficient of $(x+y)^{2n}$ is $\binom{2n}{n+k}$, where k is the number of the term counting from the middle term, and the sum to be evaluated for even values $q=2p$ is

$$\sum_{k=-n}^{n}\binom{2n}{n+k}k^{2p}. \tag{1}$$

It is a familiar theorem that when $p=0$ this sum is 2^{2n}. It proves that the method of differences can be used here provided that it be applied to the above series divided by 2^{2n}. Let

$$S(2n,q)\equiv\frac{1}{2^{2n}}\sum_{k=-n}^{n}\binom{2n}{n+k}k^{q}. \tag{2}$$

* Kenyon: *Proc. Indiana Academy of Science*, 1914.

† Chrystal: Algebra, Part II, p. 183.

‡ A paper read before the American Mathematical Society, September 5, 1918; this BULLETIN, December, 1919.

On replacing n by $n-1$ the first difference is

$$\sum_{k=-n}^{n}\left[\frac{1}{2^{2n}}\binom{2n}{n+k}-\frac{1}{2^{2n-2}}\binom{2n-2}{n+k-1}\right]k^{2p},$$

(the first and last terms in the second part of the summation being zero); this reduces to

$$\sum_{k=-n}^{n}\frac{1}{2^{2n}}\binom{2n}{n+k}\left[1-\frac{4(n+k)(n-k)}{2n(2n-1)}\right]k^{2p}$$

and hence to

$$\frac{2}{n(2n-1)}S(2n, 2p+2)-\frac{1}{2n-1}S(2n, 2p).$$

Writing the first difference as

$$D(2n, 2p) \equiv S(2n, 2p)-S(2n-2, 2p),$$

the recursion formula can be written in either of two forms

$$(3)\quad S(2n, 2p+2)=\frac{n}{2}S(2n, 2p)+\frac{n}{2}(2n-1)\,D(2n, 2p),$$

$$(4)\quad S(2n, 2p+2)=n^2S(2n, 2p)-\frac{n}{2}(2n-1)\,S(2n-2, 2p),$$

where in particular by the elementary theorem referred to above

$$(5)\qquad S(2n, 0)=1.$$

The recursion formula and (5) give the results, after multiplying by 2^{2n}:

$$(6)\quad \begin{aligned}\sum_{k=-n}^{n}\binom{2n}{n+k}k^2&=\frac{n}{2}2^{2n},\\ \sum_{k=-n}^{n}\binom{2n}{n+k}k^4&=\frac{n(3n-1)}{4}2^{2n},\\ \sum_{k=-n}^{n}\binom{2n}{n+k}k^6&=\frac{n(15n^2-15n+4)}{8}2^{2n},\\ &\cdots\cdots\cdots\cdots\end{aligned}$$

From these examples it may be inferred that the leading term in $S(2n, 2p)$ is $1\cdot3\cdot5\cdots(2p-1)\,(n/2)^p$, and this can

be proved by using (3) or (4), or as follows. Since $S(2n-2p, 2p)$ is obtained by replacing n in $S(2n, 2p)$ by $n-1$, Maclaurin's theorem gives

$$S(2n-2, 2p) = S(2n, 2p) - S'(2n, 2p) + \tfrac{1}{2}S''(2n, 2p) - \cdots$$

or

$$D(2n, 2p) = S'(2n, 2p) - \frac{1}{2!}S''(2n, 2p) + \cdots,$$

the accents indicating differentiation with respect to n; hence if the leading term of $S(2n, 2p)$ is assumed to be of degree p, that of $D(2n, 2p)$ will be of degree $p-1$ and will be the leading term of $S'(2n, 2p)$. By (3) the leading term of $S(2n, 2p+2)$ is

$$\frac{n}{2}\cdot 1\cdot 3\cdot 5\cdots(2p-1)\left(\frac{n}{2}\right)^p + \frac{n}{2}\cdot 2n\cdot 1\cdot 3\cdot 5\cdots(2p-1)\cdot\frac{p}{2}\left(\frac{n}{2}\right)^{p-1}$$

or $1\cdot 3\cdot 5\cdots(2p+1)\,(n/2)^{p+1}$; whence the theorem follows by mathematical induction. We have thus proved that the leading part of the sum (1) is $1\cdot 3\cdot 5\cdots(2p-1)\,(n/2)^p\,2^{2n}$, the chief result of this section so far as concerns the properties of binomial coefficients.

With this result the evaluation of the series of integrals dealt with in a previous paper* is exhibited thus, employing the notation used at that time and observing the usual considerations of uniform convergence:

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-x^2/2\sigma^2}x^{2p}\,dx &= \lim_{n=\infty}\sum_{k=-n}^{n}\frac{\sqrt{\pi n}}{2^{2n}}\binom{2n}{n+k}(k\Delta x)^{2p}\Delta x\\ &= \sqrt{\pi}\lim_{n=\infty}\sqrt{n}S(2n, 2p)\cdot\frac{\sigma^{2p+1}}{(n/2)^{(2p+1)/2}}\\ &= \sigma^{2p+1}\sqrt{2\pi}\,1\cdot 3\cdot 5\cdot\cdots(2p-1).\end{aligned}$$

In the language of statistics one may from this say, for example, that the mean square deviation of the area under the curve $y = e^{-x^2/2\sigma^2}$ is σ^2; in other nomenclature we may say that the second, fourth, $\cdots$ moments of this area are σ^2, σ^4, $\cdots$. It is believed that the method of evaluating this already known integral is new.

*L. c.

§ 2.

For the case where $q = 2p + 1$, we sum in like manner

$$(7) \qquad R(2n, 2p+1) \equiv \frac{1}{2^{2n}} \sum_{k=0}^{n} \binom{2n}{n+k} k^{2p+1},$$

inasmuch as considerations of symmetry show $S(2n, 2p+1)$ to be zero. The method of § 1 gives the recursion formula

$$(8) \quad R(2n, 2p+3) = \frac{n}{2}(2n, 2p+1) + \frac{n}{2}(2n-1)\,\bar{D}(2n, 2p+1),$$

where

$$\bar{D}(2n, 2p+1) \equiv R(2n, 2p+1) - R(2n-2, 2p+1).$$

Now

$$R(2n, 1) \equiv \frac{1}{2^{2n}}\left[0 \cdot \binom{2n}{n} + 1 \cdot \binom{2n}{n-1} + \cdots + (n-2) \cdot \frac{2n(2n-1)}{2!} + (n-1) \cdot 2n + n \cdot 1\right].$$

The sum of the last two terms enclosed in brackets is $n(2n-1)$, the sum of the last three terms is $\frac{n(2n-1)(2n-2)}{2!}$, etc., and mathematical induction proves that the sum of all these terms is $\frac{n}{2}\binom{2n}{n}$, i.e., $n/2$ times the first coefficient. Thus,

$$(9) \qquad R(2n, 1) = \frac{1}{2^{2n}} \frac{n}{2} \binom{2n}{n}.$$

Formulas (8) and (9) enable us to write the successive values

$$\sum_{k=0}^{n} \binom{2n}{n+k} k = \frac{n}{2}\binom{2n}{n},$$

$$\sum_{k=0}^{n} \binom{2n}{n+k} k^3 = 2\left(\frac{n}{2}\right)^2 \binom{2n}{n},$$

$$\sum_{k=0}^{n} \binom{2n}{n+k} k^5 = 2\left(\frac{n}{2}\right)^2 (2n-1)\binom{2n}{n},$$

$$\sum_{k=0}^{n} \binom{2n}{n+k} k^7 = 2\left(\frac{n}{2}\right)^2 (6n^2 - 8n + 3)\binom{2n}{n},$$

. . . . . .

By mathematical induction it is proved that the leading term in

$$\sum_{k=0}^{n}\binom{2n}{n+k}k^{2p+1}$$

is

$$\frac{1}{2}p!\,n^{p+1}\binom{2n}{n}$$

Corresponding to the application made at the end of § 1, we have here

$$\int_0^\infty e^{-x^2/2\sigma^2}x^{2p+1}dx = \lim_{n=\infty}\frac{1}{\binom{2n}{n}}\sum_{k=0}^{\infty}\binom{2n}{n+k}(k\Delta x)^{2p+1}\Delta x$$

$$= \lim_{n=\infty}\frac{1}{2}p!\,n^{p+1}\left(\frac{2\sigma^2}{n}\right)^{p+1}$$

$$= \frac{1}{2}p!\,(2\sigma^2)^{p+1}.$$

For example, since the area under the whole curve $y = e^{-x^2/2\sigma^2}$ is $\sigma\sqrt{2\pi}$, the "mean deviation" of this area is $\sigma\sqrt{2/\pi}$.

The products of the binomial coefficients by powers of terms of other arithmetical progressions do not seem to give simple results analogous to those obtained by Kenyon; this question is reserved for further study.

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THE WORK OF POINCARÉ ON AUTOMORPHIC FUNCTIONS.

Oeuvres de Henri Poincaré, publiées sous les auspices du Ministère de l'Instruction publique par G. DARBOUX. Tome II, publié avec la collaboration de N. E. NÖRLUND et de ERNEST LEBON. Paris, Gauthier-Villars, 1916. lxxi + 632 pp.

THE collected works of Poincaré will fill some 10 volumes, of which the one before us is the first to be published. It contains the principal papers written by him in the field of

automorphic functions, where some of his most brilliant early work lies. According to the statement of Darboux in the preface, this Volume II appears first with the hope that it may stimulate mathematicians to active work in that field.

An invaluable and necessary revision with critical notes is supplied by Nörlund.

The many remarkable eulogies pronounced shortly after the death of Poincaré testify to a very wide recognition of his dominating position in the mathematical world. In one of the best of these, Volterra says: "if we were to characterize the recent period of the history of mathematics by a single name we should all give that of Poincaré."* Among these appreciations that of Hadamard† may be mentioned here for its critical value, while the admirable "Eloge Historique" of Darboux deserves its introductory place in the volume.

It is impossible to give any satisfactory idea of the achievements of Poincaré in the space of a single essay. The appearance of his collected papers arranged by subjects will furnish an occasion for a more leisurely and critical review of his work in its relation to the mathematics of the time. It is my purpose to attempt such a review.

An immediate stimulus for Poincaré's researches in the domain of automorphic functions was the question proposed for the Grand Prix of 1880 in the mathematical sciences: to complete in some important point the theory of ordinary linear differential equations. Partly on account of incompleteness in his development of the new functions, the prize was not awarded to Poincaré but to G. H. Halphen. An extract from this initial attempt of Poincaré is to appear in volume 39 of the *Acta Mathematica*.

In order to understand the nature of the advance made by Poincaré it is necessary to go back to the nearly contemporary work of Schwarz, Fuchs, Schottky, and Klein.

It was obvious after Riemann's time that the inverse of the ratio of a pair of solutions of an ordinary linear differential equation of the second order was an automorphic function, i.e., one unaltered by a group of linear fractional transformations. Examples of such functions were at hand

*Volterra, "Henri Poincaré." A lecture delivered at the inauguration of Rice Institute. Translated by G. C. Evans. Rice Institute Pamphlets, Vol. I (1915), No. 2, pp. 133–162.

†"Henri Poincaré: le mathématicien," *Revue de Métaphysique et de Morale*, vol. 21 (1913), pp. 617–658.

in the trigonometric and elliptic functions, the elliptic modular functions, the more general triangle functions first classified by Schwarz, other functions due to Schottky, and finally the rational automorphic functions whose beautiful algebraic and geometric relations were developed by Klein. When considered in relation to the illuminating work of Fuchs on ordinary linear differential equations, these examples made it an obvious probability that an extensive theory of single-valued transcendental automorphic functions awaited development. Fuchs pointed out this field explicitly first in 1880, but Poincaré's account of the genesis of his ideas indicates his essential independence of this paper.*

The general transcendental automorphic functions, however, were not immediately approached, because of a certain gap to which we will now refer.

If we take the singular points of the differential equation and the characteristic exponents as real, it is an obvious deduction from the work of Fuchs before 1880 that the inverse function maps a circular polygon conformally upon a Riemann's surface, and that by the process of analytic extension other polygons which are obtained by linear fractional transformations, arise. An immediate necessary condition that there is no overlapping of the polygons is that the angles at the vertices of the first polygon are either 0 or fractional parts of 2π. The precise further conditions for non-overlapping were known only in the case of the rational automorphic functions, treated by Klein, where the polygons were representable as spherical polygons so that the ordinary formulas of spherical trigonometry were available. But the earlier work of Fuchs made it clear that single-valued automorphic functions exist always when there is no overlapping.

Thus the moment at which Poincaré reached scientific maturity was a most favorable one in which to create a general theory of these functions which were challenging the attention of mathematicians.

Here it seems worth while to call attention to a situation which often exists in mathematics but scarcely ever exists in any other field of science. It is sometimes the case that several mathematicians are in possession of much new material save for a single link of reasoning. However, on

* "Science et Méthode," Flammarion, Paris, 1908, Chap. 3; or pp. lvii–lviii of Darboux's "Éloge Historique."

account of self-imposed requirements of completeness and rigor, no claim is made for credit.

It was precisely such a state of affairs that existed in the domain of the transcendental automorphic functions when Poincaré began his work. The algebraic relation between automorphic functions with the same group, the uniformizing properties of these functions, and their use in solving linear differential equations are illustrations of material of this sort.

It was by his use of non-euclidean geometry that he overcame the one fundamental difficulty referred to above. How then did these geometric ideas enter?

The region of the complex plane under consideration is the interior of a fixed circle and the bounding circles of the polygons referred to are all orthogonal to it. Furthermore, the linear fractional transformations of these polygons are such as to leave the fixed circle invariant. Here, then, are the abstract features of the non-euclidean geometry of Lobachevski, namely, a three-parameter group of transformations which may be interpreted as the group of rigid motions if every circle orthogonal to the fixed circle be regarded as a straight line, and if angles be interpreted as usual.

In recognizing this fact Poincaré faced an extensive vista of possible developments in the field of automorphic functions, and incidentally obtained a simple new geometric representation of the hyperbolic plane. It was natural to connect this geometric view with the highly suggestive work of Klein. Thus we are not surprised to find Poincaré a master of the powerful weapons furnished by projective geometry at the outset.

From this new point of view the difficulty of overlapping presents an absolutely concrete aspect. The polygon appears as an ordinary polygon in the non-euclidean plane, and the problem is to determine when the entire plane can be filled by a network of congruent polygons. The precise analytical conditions are seen to be algebraic.

Although the essential kernel of the advance of Poincaré was only a single step, he must be regarded as the true founder of the theory of the general transcendental automorphic function. A somewhat analogous situation is presented by the founding of the calculus. Without doubt many mathematicians were in possession of most of the essential ideas of

the calculus, but failed to discern the possibility of elevating it to an independent discipline by the invention of a suitable symbolism. Nevertheless there is general agreement that Newton and Leibniz were the originators of the calculus, precisely because they did see this possibility.

Had Newton merely developed the simpler parts of the calculus of fluxions, his title to preeminence would have lost some of its validity. But Newton not only conceived the possibility but carried it to fulfilment by developing an adequate technique and applying it to the problems of celestial mechanics. Likewise, Poincaré merits great credit because, once in possession of the fundamental new weapon of attack, he proceeded to develop the theory of the transcendental automorphic functions to a remarkable degree.

In the further development of his new ideas Poincaré faced two possibilities. On the one hand, following the general course of Riemann, Schwarz, Fuchs, Schottky, and Klein, he might develop a general theory based on existence theorems for conformal mapping. This was a possible course as he saw plainly. Thus he declares in an early paper (*Mathematische Annalen*, 1882, translation) "From the existence of discontinuous groups one might without doubt, by processes analogous to those of Schwarz, deduce that of single-valued functions reproduced by the substitutions of the group; but one would have then no explicit expression for these transcendental functions. Moreover it is better to use other methods." The alternative method was based upon certain explicit series invented by Poincaré before he noted the geometric relation outlined above, and to these series we now turn.

In the case of an ordinary finite group of linear fractional transformations the sum of a rational function $H(z)$ of z, taken for any value of z and its transformed arguments z_k under the group, clearly forms an invariant function. Similar infinite series exist in the case of the series for the elliptic functions. These infinite series, invented by Cayley and Eisenstein, diverge in certain cases although their derivative series converge. On the basis of such series Weierstrass constructed his theory of the elliptic functions.

Suppose that in an analogous way we form the series formally invariant under an arbitrary infinite discontinuous group of linear fractional transformations leaving invariant a circle. The image points z_k of z will approach the invariant circle,

as k becomes infinite, and thus the kth term $H(z_k)$ of the series will remain finite, at least if the rational function H with which we start has no poles upon that circle. But there is no reason to assume that this series converges.

If, however, the series be formally differentiated term by term, the typical terms of the new series are the product of the derivative rational function $H'(z_k)$ taken at the image point z_k and the derivative dz_k/dz. The modulus of the second factor represents the ratio of arc lengths of corresponding similar infinitesimal figures. The area integral of the squared modulus of this factor over any part C of a fundamental region yields the ordinary area of the image C_k. But the sum of all the areas C_k is less than the area of the invariant circle. Hence the series

$$\Sigma \int\int \text{mod } (dz_k/dz)^2 dx dy$$

converges. An easy extension of this argument shows that $\Sigma \text{mod}(dz_k/dz)^2$ also converges. Thus the factor dz_k/dz approaches 0 as k becomes infinite.

Now if such a derivative series converges it does not represent an automorphic function but a function I with the property

$$I(z_k) = I(z)(c_k z + d_k)^{-2},$$

where

$$z_k = (a_k z + b_k)/(c_k z + d_k)$$

with $a_k d_k - b_k c_k = 1$. Thus one is led to consider series whose typical term is of the form $H(z_k)\ (dz_k/dz)^m$ which will converge if $m \geqq 2$, and which are multiplied by $(c_k z + d_k)^{-2m}$ when z is changed to z_k.

Moreover the functions defined by these theta series of Poincaré play the rôle of the elliptic theta functions for elliptic functions and are suited to form a basis for the general development of the theory of automorphic functions; in fact the quotient of two such functions having the same value of m is an automorphic function. No other completely explicit expressions for the automorphic functions have as yet been found.

All of these facts can be looked at from the point of view of homogeneous variables and then take on a more elegant form, but it seems probable that the development of Poincaré's own ideas concerning these series was approximately along the line of least resistance just outlined.

With these two touchstones of the theory in his possession—namely, the interpretations of non-euclidean and projective geometry, and the explicit series above mentioned—the anâlogy of the theory of automorphic functions and the elliptic functions became doubly apparent. The interrelation of elliptic functions with algebraic functions and their integrals, with linear differential equations, and with number theory were seen by him to admit of complete generalization, and the range of knowledge called for in order to carry out this development was just that which Poincaré had come naturally into possession of at Paris.

His five long papers in the early volumes of the *Acta Mathematica* give a first approximation to the theory of the automorphic functions even at its present degree of development. A large variety of important complementary results have been added, mainly by German mathematicians, who have used homogeneous variables, and have developed an independent theory from the point of view of Riemann alluded to above.

In the first of these extensive papers entitled "Théorie des groupes fuchsiens," Poincaré developed explicitly the forms of polygons and networks of congruent polygons in the non-euclidean plane from his geometric point of view. He called discontinuous groups which possess an invariant circle Fuchsian groups in honor of Fuchs, and the corresponding automorphic functions Fuchsian functions. Perhaps it is better to use the terminology of Klein, and speak of groups with principal circle and automorphic functions with principal circle. Poincaré's classification of the types of groups is suggestive but has not the definitive form which has been given it by Fricke.

In a second paper "Sur les fonctions fuchsiennes" he considers the series above obtained and develops the main facts concerning the functions which they define. The chief difficulty is caused by the fact that such a function may vanish identically. By means of the theory of these functions a theory of the automorphic functions can also be derived. A cardinal fact to prove here is that every automorphic function can be represented as the quotient of two such series. Conversely, one can pass from the automorphic function back to the theta series of Poincaré by a process of differentiation.

A first fundamental application of these functions, noted in the same article, is to the uniformization of algebraic functions. Between any pair of automorphic functions with the same group an algebraic relation obviously exists, just as an algebraic relation exists between two doubly periodic functions with the same period parallelogram. The automorphic functions uniformize this algebraic relation in the sense that each variable is expressible by means of an automorphic function belonging to the same group. In this way the Riemann surface is mapped conformally upon the circular polygon referred to earlier. Thus the important question arises at once: is it not possible to uniformize every algebraic relation by means of such function? An early count of constants showed Poincaré that this was in all probability the case. A second application of importance lay in the integration of the corresponding linear differential equation of the second order.

In a third paper "Mémoire sur les groupes kleinéens" Poincaré treated the more general automorphic group in which there was no invariant circle, when the same theta series were available. These groups and the corresponding automorphic functions were called Kleinian by him, but are perhaps better designated as groups and automorphic functions without principal circle. The fundamental geometric results which made an attack on these more general functions possible were due to Cayley and Klein. Spatial non-euclidean geometry enters, inasmuch as the totality of linear fractional transformations of the complex plane is isomorphic with those projective transformations of a quadric in space which leave the quadric invariant.

In two further papers, "Sur les groupes des équations linéaires" and "Mémoire sur les fonctions zétafuchsiennes," Poincaré attacks the problem of uniformizing algebraic functions and the integrals of linear differential equations with algebraic coefficients. It is easy to see that if the function has only three branch points or if the linear differential equation has rational coefficients and three singular points, regular or irregular, then such a uniformization is possible by means of the elliptic modular function, provided the images of the three singular points lie at the vertices of the fundamental triangle. This admits of obvious generalizations. In his attempt to establish such generalizations he proves first that

the characteristic constants of the monodromic group are entire functions of the literal coefficients, and then employs the "method of continuity" of Klein. The satisfactory application of this method has proved difficult. Poincaré gave explicit series analogous to his theta series in terms of which such uniformization for the differential equation is possible.

In attempting to appraise the importance of the new functions discovered by Poincaré, one must always remember that these functions form a natural extension of many of the functions treated earlier in analysis. Moreover, they afford a very simple systematic method of approach to the theory of the algebraic functions and their integrals, and add an illuminating chapter in the theory of ordinary linear differential equations with algebraic coefficients. But the process of natural generalization appears to terminate with the invention and investigation of these functions, just as the theory of the elliptic functions reached its conclusion earlier. I do not expect to see large further developments take place, notwithstanding the hope to the contrary implied by Darboux in the preface.

The mathematician of the future, having before him such completed theories, will seek on the one hand to clarify and simplify, and on the other hand to use only as much of them as is really vitally related to further advance.*

If this be true it may be predicted that the elliptic modular functions and the triangle functions will maintain an important position on account of their intrinsic importance and as the best examples of transcendental automorphic functions with principal circle, while the more general functions discovered by Poincaré will receive only such consideration as is necessary to establish their position in the hierarchy of analytic functions.

GEORGE D. BIRKHOFF.

* A treatment of automorphic functions in this spirit is given in the concluding chapter of Professor Osgood's "Lehrbuch der Funktionentheorie," vol. 1 (second edition), Teubner, Leipzig, 1912. See also his paper "On the uniformization of algebraic functions," *Annals of Mathematics*, ser. 2, vol. 14 (1912–1913), pp. 143–162.

A BRIEF ACCOUNT OF THE LIFE AND WORK OF THE LATE PROFESSOR ULISSE DINI.

BY PROFESSOR WALTER B. FORD.

(Read before the American Mathematical Society September 3, 1919.)

THE death of Professor Ulisse Dini in his native city of Pisa, Italy, on the 28th of last October marked the passing of one whose name has long been familiar to the mathematical fraternity of the entire world and it therefore seems fitting, despite the wide separation geographically between the scene of his labors and our own, that a brief account of his life and work be given at this time in America and a small measure at least of tribute be rendered to his genius.

Dini was born on the 14th of November, 1845, of parents highly respected but of very moderate circumstances. He early manifested an unusual activity of both mind and body, thus commanding the attention and admiration of his instructors and masters who foresaw for him a future of extraordinary promise. Upon entering the neighboring university of Pisa his marked capabilities in mathematics were soon recognized and he shortly became the favorite pupil of Professor Betti, well known as one of the leaders in mathematical instruction and research at this period in Italy. Under such auspices and at the uncommonly early age of nineteen years, Dini attained the laureate and received directly afterward a government scholarship enabling him to continue his studies for a year at Paris. During this brief foreign sojourn he came chiefly under the influences of Bertrand and Hermite and formed an intimate friendship with them which extended long into later years. Returning forthwith to Pisa, his active career as teacher and investigator began in the year 1866 while he was yet scarcely of age and continued almost without interruption for over fifty years, being characterized throughout by the utmost zeal and devotion, not only to mathematical science, but to his native country, city and university, from all of which he received in turn the highest honors they could bestow.

Such in brief was Dini's career. If we inquire in a more detailed sense what his relations to mathematics actually

were and what his special achievements, it may be noted in the first place that his earliest researches lay in the field of infinitesimal geometry; more specifically, in the determination of the form and properties of certain partial differential equations which arise in the theory of applicable surfaces. His work in this connection, though extending over no more than six years (1864–1870), gave rise to some eighteen memoirs dealing chiefly with general problems in the theory of curvature and geodesics, some of which had been proposed earlier by Beltrami. At this early period of his life, however, Dini had not yet begun the researches for which he is to be regarded as famous nor had he in fact even entered seriously into that broad field, namely pure analysis, in whose development he was destined soon to play an active part. The transition of his interest and labors to this latter field took place about 1870. At this comparatively early date it will be recalled that the newer and more rigorous analysis (the so-called "modern analysis" of to-day) was but little known to the world at large, the spirit of its methods being virtually confined to the limited school of pupils immediately surrounding Weierstrass in Germany. Nevertheless, once Dini had turned his efforts in this direction, he appears to have reached within a remarkably short time a full appreciation of these newer ideas and methods. In fact, he straightway acquired such a critical insight into their significance and developed such ability and confidence in their use that he was soon independently at work carrying out for himself their manifold consequences, especially their bearing upon those concepts which lie at the foundation of analysis. Thus, by the year 1877, or seven years from the time he began, he published the treatise, since famous, entitled Foundations for the Theory of Functions of Real Variables (Fondamenti per la teoria delle funzioni di variabili reali). Much of what Dini here sets forth concerning such topics as continuous and discontinuous functions, the derivative and the conditions for its existence, series, definite integrals, the properties of the incremental ratio, etc., was entirely original with himself and has since come to be regarded everywhere as basal in the real variable theory. The book has, in fact, served as a model the world over and even at this date, which is more than forty years since its publication, it still affords one of the best available expositions of the basal concepts of analysis

as regarded from the standpoint of modern rigor, evidence of which may be found, for example, in the fact that as late as 1902 an authorized translation of the work was published in German.

The fact that Dini, though situated at the time far from any natural sources of information, was able to arrive so early at a thoroughgoing appreciation of the newer methods in analysis and to produce straightway so unique and valuable a work as the "Foundations" leads one naturally to inquire somewhat farther than we have thus far indicated into the attending circumstances. One may well ask, for example, what particular problem or incident served as his starting point and thus became his first inspiration. Dini's own explanation of this, as given in after years to inquiring friends, was somewhat as follows: While still a pupil at the university he came to feel serious doubts as to the actual meaning of some of the most fundamental concepts commonly employed in the calculus. They did not seem to be capable of the sort of definite, working formulation which one has a right to expect of every concept employed in mathematical science. While thus concerned in his own mind, he learned some years later from certain memoirs of Hankel, Schwartz, and Heine that similar doubts had been experienced by others and had already been resolved, at least in part, by Weierstrass and his school in Germany. Thereupon he obtained from this distant source such further information as was then available and, though meager and confined to special problems only, it was sufficient to impart the spirit of the newer methods and to convince him of their far reaching possibilities and consequences. He then began, single handed and alone, that remarkable series of studies and researches which culminated but a few years later in his treatise above mentioned on the Foundations.

With the assurance once gained that he was working upon well-grounded principles and definitions, Dini next proceeded to apply them in an extended and detailed sense. Thus, during 1877–78 he reworked and treated in his lectures a wide variety of topics taken from the usual course in higher analysis. In particular, he here gave for the first time a rigorous treatment of the general theory of implicit functions. His numerous researches at this period were left unpublished, however, being preserved only in lithograph form. Not

until the later years of his life did he undertake the considerable task of arranging the whole for publication, but it may now be found, together with much supplementary material, in his four large volumes published as late as 1915 entitled Lessons in Infinitesimal Analysis (Lezioni di analisi infinitesimale).

Following the studies of a very general nature above referred to, Dini turned his attention in 1878, or thereabouts, to a more specific field; namely, to a critical investigation of the convergence and other important properties of Fourier series. The result appeared two years later in the form of another treatise, this being entitled Fourier Series and other Analytical Representations of the Functions of a Real Variable (Serie di Fourier e altre rappresentazioni analitiche delle funzioni di una variabile reale). Here we meet with an entire book of 328 pages representing a single piece of sustained, original research and embracing results of a remarkably high order. Not only is the convergence of Fourier series studied from many points of view, but a complete and satisfactory discussion is here given for the first time of the convergence of various other allied developments well known in mathematical physics, such as the developments of an arbitrary function in terms of Bessel and Legendre functions. Moreover, a genera' method is finally set up for the examination of all series o the Fourier type (series in terms of normal functions). Dini': wonderful powers, both critical and inventive, doubtless reacl their highest realization in this work, though, unlike th Foundations, the style of presentation is not well calculatec to bring out the merit of the whole. This feature is largely due, as noted in the preface, to the fact that the publicatior proceeded at intervals in such a way that no opportunity was afforded to put the earlier parts of the work in proper adjustment with the later ones.*

* As the development of an arbitrary function in terms of normal functions has been extensively considered in recent years, especially from the standpoint of the modern theory of integral equations, a word should perhaps be said as to the relation between these more recent studies and those of Dini above mentioned. Taking the question of convergence alone, the results of the more recent studies may be said to reach their culmination in the researches of Haar, *Math. Ann.*, vol. 69 (1910), p. 331 and vol. 71 (1911), p. 38. These, while exceedingly general, do not, however, include such important special cases as the Bessel and Legendre developments mentioned above. For a summary statement of Haar's results see Bôcher's address at the international congress of mathematicians at Cambridge (England), 1912, pp. 29–30, and for the underlying reason to account for the limitations which such results present, see the same address, p. 1,

Thus far we have mentioned only Dini's larger treatises. Space would hardly permit of a detailed consideration of his shorter memoirs, and fortunately this is not necessary, since the list, which comprises in all about sixty papers, has recently been prepared by his colleague, Professor Luigi Bianchi, and published along with a more extended account of his entire scientific achievements in the proceedings of the Accademia dei Lincei of last February. Suffice it to say that these memoirs pertain largely to infinitesimal geometry, to the theory of functions of a complex variable and to the study of differential equations, total and partial.

We might close our brief account at this point were it not for the fact that an adequate conception of Dini's activities can scarcely be gained from an examination of his scientific career alone. Dating from about the year 1880, he was chosen time and again to occupy positions of honor and trust in the affairs of his city, province and nation. In particular, he was at the time of his death vice-president of the national council of public instruction, a senator of the kingdom and the director of the normal college (Scuola Normale Superiore) of Pisa, in addition to his position as professor at the university. His death, therefore, was deeply mourned by a wide circle of people and institutions all of which have since united in an effort to erect a permanent monument within the city of Pisa that shall symbolize not only his genius in science but the love and esteem in which he was universally held by his countrymen.

UNIVERSITY OF MICHIGAN.

SHORTER NOTICES.

Commercial Algebra. By WENTWORTH, SMITH and SCHLAUCH. Boston, Ginn and Company, 1917–1918.

THIS work appears in two volumes, Book I designed for the first year course of a commercial high school, and Book II a text for advanced classes in the same lines.

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statement (4) at the bottom of the page. The excluded cases of importance have usually been considered (though not by Dini) only through special methods adapted to the case in hand. Thus, see Hobson, *Proc. London Math. Soc.*, vol. 7 (1908), pp. 359–388; also C. N. Moore in various articles in the *Trans. American Math. Soc.* dating from 1907.

As compared with the usual first book in algebra, there are a few departures in Book I. The authors call attention in the preface to the omission of "a large amount of non-essential work in factoring, fractions, and complicated equations." The problems, too, are chosen with a view to familiarizing the student with commercial phraseology and presenting appropriate applications to such topics as shop formulas, daily balances, cost-rate factors, commission formulas, etc. One of the newer features is the application of simple equations to problems in standard dietaries and ratio of solvency. Graphic work is introduced in elementary form in Chapter III, and considerable practice is given in pictograms and cartograms, with special reference to advertising and statistics. In Chapter XIII a more extended treatment of graphic methods is given, including the topics of wage graphs, interpolation, net profit, and interest.

In Book II the first four chapters present the topics of algebra necessary as a preparation for the applied work, namely, powers and roots, logarithms, slide rule, and series. The remainder of the book treats both theoretically and practically the subjects of compound interest, equation of payments, annuities, amortization, depreciation, bonds, life insurance, and alignment charts. This volume contains tables of compound interest, present value, annuities, and logarithms.

While these volumes follow rather closely the lines of texts already on the market, both in content and method of presentation, a particular point has been made of so simplifying the work as to adapt it to the needs of the high school. This adaptation appears in the omission of the more advanced work in series, the choice of problems, and the occasional replacing of abstract notation by significant letters, as in the formulas connecting nominal and effective rates of interest.

The two books are a useful contribution to the organization of high-school courses, in that they meet present-day demands for practical topics, without sacrificing an intelligent understanding of the tools used—an end which can be gained only by a careful development of the theoretical side.

The publication of such texts as this may be taken as a hopeful sign—not that the curriculum is becoming over commercialized, but that the commercial world is finding it

increasingly necessary that its people shall be trained in logical mathematical thought.

F. E. ALLEN.

The Training of Teachers of Mathematics for the Secondary Schools of Countries represented in the International Commission on the Teaching of Mathematics, by RAYMOND CLARE ARCHIBALD. With the Editorial Cooperation of D. E. SMITH, W. F. OSGOOD, J. W. A. YOUNG, members of the Commission from the United States. Washington, Government Printing Office, 1918.

THIS well-executed piece of work includes an able discussion of the training of teachers in Australia, Austria, Belgium, Denmark, England, Finland, France, Germany, Hungary, Italy, Japan, the Netherlands, Roumania, Russia, Spain, Sweden, Switzerland, and the United States, as well as appendices with typical examinations as given in England, France, Germany, and Japan. The Summary is also a valuable feature.

No one can deny the assertion that most of the countries discussed have far higher standards for their teachers of mathematics in secondary schools than we have in the United States. In this connection the interesting fact is brought out that there were in the United States, in 1913–1914, 13,714 "accredited secondary schools." Similar statistics for other countries, unfortunately not given, would have been of great significance.

Our need for well-trained teachers is not less than that of any other country. However, our problem to obtain and train these teachers has been fundamentally different from the European problem of obtaining teachers. We have been trying to educate all the children of all the people, whereas no great European country has made any similar attempt. England, France, Germany, Italy and Russia are all on the same footing, in training only a select few and differentiating *upon entrance into the most elementary school* the elect, those who may go on, from those who will not go on into secondary schools. With this system, the relatively few teachers needed can be given a higher and more intensive training.

The effect of the war upon the educational systems of European countries will be great. Undoubtedly the present industrial unrest will culminate in the democratization of the

schools of all the world. More than ever it is incumbent upon American leaders in education to exert every effort to improve the preparation of the teachers and the status of the profession, for it is to our system of education that Europe will turn for guidance.

The recent tremendous increases in enrollment of students in our universities includes fortunately large numbers who are taking up the study of mathematics. It is somewhat significant that this increase is affecting only the universities, not the normal schools. Undoubtedly, the day is not distant when our universities and colleges will be called to train not only high school teachers, but also the "junior high school" teachers. The nature of this training for the instructors in mathematics in the secondary schools needs systematic and continued study, taking into account the changes in the world in which we live.

This study under review, while reflecting a world which has passed, will nevertheless continue for many years to be of great value to all concerned with the preparation of our teachers of mathematics.

L. C. KARPINSKI.

Theory of Maxima and Minima. By HARRIS HANCOCK. New York, Ginn and Company, 1917. v + 193 pages. Price $2.50.

THE student of mathematics meets the interesting subject of maxima and minima early in his first course in calculus. The subject appears again and again in his courses in advanced calculus and functions of a real variable, each time carrying the student a little deeper into the theory, but rarely giving him opportunity to view the subject as a whole. The English reading student finds it particularly difficult to study the theory for functions of two or more variables as expounded by Scheefer, Stolz, von Dantscher, and Weierstrass. Professor Hancock's book is for this English reading student.

The opening chapter discusses functions of one variable, first taking up functions having complete derivatives throughout the interval in question (ordinary maxima and minima). This covers the usual discussion in a first course in calculus with a somewhat more mature reader in view. Then follows a discussion for functions having derivatives only for definite values of the variable, or having one-sided derivatives (ex-

traordinary maxima and minima). A statement of the theorem concerning the existence of the upper and lower limits of a continuous function in a definite interval is included without proof.

Chapter II is a preliminary chapter on functions of several variables, outlining in a broad way the general problem. Chapter III is limited to functions of two variables, treating in a general way the different cases that may arise, pointing out the difficulties of the ambiguous case, discussing the errors that are in the literature and noting the various attempts to improve the theory. This leads up to Chapter IV in which Scheefer's theory for two variables is developed in detail with a general outline of Stolz's extension to the case of three variables. Von Dantscher's theory is included for comparison purposes. These three chapters are the most important in the book. They give in sufficient detail the theorems of Scheefer, Stolz and von Dantscher together with a discussion of the errors made by earlier writers and carried on into many of the modern textbooks on the calculus. In particular the so-called ambiguous case, around which most of the difficulties center, is treated in a fuller manner than in any other work in English that the reviewer has examined. On page 34, the author quotes from Professor Pierpont's article in Volume IV of the BULLETIN, "Our English and American authors seem to be ignorant of these facts." At first the reviewer was inclined to look upon this quotation written in 1898 as rather historical, but a study of the English and American books available failed to bring forth a full discussion of the maxima and minima of functions of two variables. However in many cases the reason was not that the authors seemed "to be ignorant of these facts," but simply that they lacked the space for a full treatment. This full treatment is here.

The last four chapters in the book follow very closely the work published by Professor Hancock in 1903, Lectures on the Theory of Maxima and Minima of Functions of Several Variables (Weierstrass' Theory).

These chapters, which follow the lectures of Weierstrass, treat only the ordinary cases where the functions are everywhere regular and where the forms are either definite or indefinite, excluding the semidefinite form characterizing the ambiguous case. Some interesting additions have been made

to the older work. Among them are two interesting light theory problems prepared by Professor Brand. One problem is concerned with the necessary and sufficient conditions for minimizing the length of a ray of light from one point to another after reflection at a point on the surface $F(x, y, z) = 0$. The other is a refraction problem in which the time between two points is to be a minimum. Hadamard's determinant problem is also included.

Throughout the book are many examples and problems illustrating points in the theory, some worked out in detail.

A. R. CRATHORNE.

From Nebula to Nebula, 4th Edition. By GEORGE HENRY LEPPER. Pittsburgh, Pa., privately printed, 1919. 401 pages.

(FROM the introduction) "The object of this work is two-fold: (1) to present a revaluation of the time-honored doctrines upon which modern theoretical astronomy is based, and (2) having shown wherein they are defective, to propose a new and far more comprehensive system, revealing the entire visible universe in the philosophic aspect of a single unit coordinated throughout, as *a priori it must be*, by a single dynamical force."

It is not possible, within the limits of available space, to give a complete analysis of this book. A few quotations will indicate some of the author's conclusions, but the proofs which he gives must be omitted. These quotations are so chosen that the lack of context need do no injustice to the author.

"(The earth . . .) is generously endowed with liquid oceans, which she is compelled to shift constantly from the sunlit side in her efforts to preserve her center of gravity at the lowest point. It is this struggle for equilibrium . . . which I assign as the major cause of the earth's diurnal rotation."

"I contend, with Aristotle, that rest is the natural state of matter."

"I have been enabled . . to demonstrate the sun's path to be that of an immense spiral with a diameter of 1,530,000 million miles and a total coil length of nearly seven trillions."

"Paradoxical as it may sound, . . . I contend that the sun and not the moon is the *vera causa* of the tides."

After attempting to prove that tides cause the rotation of

the earth the author states: " . . . it becomes quite evident that the earth is not turning on its axis simply 'because it can not stop a motion that never was started,' but because Nature intelligently supplies a quarter quadrillion horse power engine to do the work."

This review may well end with the following quotation which Lepper himself gives, ascribing it to the director of the Smithsonian Institute, and which is illustrated by this volume. "Every large scientific institution or observatory has almost daily communications from persons of very moderate attainments who presume to question, nay rather to spurn, the most well-attested facts of human knowledge. . . . No argument can refute them because they have not the requisite learning to comprehend it, which is no disgrace but which should make men *modest* enough to have faith in *those who excel them immeasurably*. . . ."

F. H. SAFFORD

A Treatise on the Analytical Dynamics of Particles and Rigid Bodies; with an Introduction to the Problem of Three Bodies. By E. T. WHITTAKER. Second edition. Cambridge, University Press, 1917. xii + 432 pp.

THE second edition differs from its predecessor mainly in that references to more recent researches are given together with a brief outline in certain cases.

The book is invaluable as a condensed and suggestive presentation of the formal side of analytical dynamics. There is serious need for a complementary type of treatise in which the main emphasis is laid upon the deeper qualitative side which has played an increasingly larger part since the work of Poincaré.

As a single instance of the incompleteness of Whittaker's treatment in this respect, the fact may be mentioned that in his final chapter he treats the trigonometric series precisely in the spirit of Delaunay and does not even mention that these series are generally divergent nor refer to their asymptotic properties.

G. D. BIRKHOFF.

NOTES.

THE concluding (October) number of volume 20 of the *Transactions of the American Mathematical Society* contains the following papers: "Line-geometric representations for functions of a complex variable," by E. J. WILCZYNSKI; "On the combination of non-loxodromic substitutions," by E. B. VAN VLECK; "On a general class of integrals of the form $\int_0^\infty \varphi(t)\, g(x+t)\, dt$," by R. D. CARMICHAEL; "Transformations of surfaces applicable to a quadric," by L. P. EISENHART; "Note on two three-dimensional manifolds with the same group," by J. W. ALEXANDER; "A general system of linear equations," by A. J. PELL; Indexes, volumes 11–20.

AT the Brussels meeting of the National Research Council several International Unions were formed. For other unions, where the representation was insufficient divisional committees were formed. In mathematics a divisional committee was nominated and charged with the duty of circulating a draft of statutes so as to obtain the opinion of the principal bodies concerned. M. DE LA VALLÉE POUSSIN was elected president, and the following secretaries of this committee: TH. DE DONDER (Brussels); G. KOENIGS (Paris); M. PETROWITCH (Belgrade); V. REINA (Rome).

The above information of action taken six months ago now reaches American mathematicians for the first time by indirect ways. Before the war international mathematical activities were carried on very efficiently by mathematicians and it may be hoped that they will soon resume charge of their affairs.

AT the meeting of the Edinburgh mathematical society on November 14, the following papers were read: By E. T. WHITTAKER, "A new basis for graduation"; by H. W. RICHMOND, "A geometrical proof of Morley's extension of Feuerbach's theorem."

THE two Benjamin Peirce instructorships in mathematics at Harvard University (see this BULLETIN, volume 21, page

315) are again open to general competition. Applications for the year 1920–21, accompanied by the necessary papers, should reach Professor Osgood not later than February 15, 1920, and all communications relating to these appointments should be addressed to him.

PROFESSOR E. CZUBER, of the Vienna technical school, has retired from active teaching.

AT the University of Hamburg, Dr. W. BLASCHKE and Dr. E. HECKE have been appointed to professorships and Dr. J. RADON to an associate professorship of mathematics.

DR. K. KNOPP has been appointed professor of mathematics at the University of Königsberg.

AT the Dresden technical school, Dr. P. E. BÖHMER has been appointed professor of the mathematics of insurance and Dr. R. VON MISES professor of the resistance of materials, hydraulics, and aerodynamics. Professor J. M. KRAUSE has retired from active teaching.

DR. R. BALDUS has been appointed professor of descriptive geometry at the Karlsruhe technical school.

DR. V. BLAESS has been appointed associate professor of mechanics at the technical school at Darmstadt.

THE following persons have recently been admitted as privat-docents in mathematics at German universities: P. BARNAYS, Miss E. NOETHER, and M. SCHMEIDLER, at the University of Göttingen; P. EPSTEIN, at the University of Frankfurt; F. LEVI, at the University of Leipzig.

THE Royal society of London has awarded a Royal medal to Mr. J. H. JEANS, for his researches in applied mathematics, and the Sylvester medal to Major P. A. MACMAHON for his researches in pure mathematics, especially in connection with the partition of numbers and analysis.

ASSISTANT professor A. F. CARPENTER, of the University of Washington, has been promoted to an associate professorship of mathematics.

PROFESSOR ARNOLD DRESDEN, secretary of the Chicago Section of the American Mathematical Society, has returned to the University of Wisconsin, after a year of service with the Red Cross in France.

AT the University of California, assistant professor B. M. WOODS has been promoted to a full professorship of aerodynamics.

AT the University of Oklahoma, Dr. H. C. GOSSARD, of the U. S. Naval Academy, has been appointed assistant professor of mathematics, and Mr. E. D. MEACHAM has been promoted to an assistant professorship.

ASSISTANT professor FLORENCE P. LEWIS, of Goucher College, has been promoted to an associate professorship of mathematics.

AT Purdue University, assistant professor GLENN JAMES has resigned to become assistant professor of mathematics at the Carnegie Institute of Technology. Assistant professor T. E. MASON has been promoted to an associate professorship. Professor LAURENCE HADLEY, of Earlham College, has been appointed associate professor of mathematics, and Professor F. H. HODGE, of Franklin College, instructor in mathematics.

ASSOCIATE professor R. B. MCCLENON, of Grinnell College, has been promoted to a full professorship of mathematics.

ASSISTANT professor C. N. MILLS, of South Dakota State College, has been appointed professor of mathematics at Heidelberg University, Tiffin, Ohio.

ADJUNCT professor J. J. LUCK has been promoted to an associate professorship of mathematics at the University of Virginia.

DR. C. H. FORSYTH, of Dartmouth College, has been promoted to an assistant professorship of mathematics.

IN the mathematics department of the U. S. Naval Academy, Dr. C. C. BRAMBLE has been promoted to an assistant pro-

fessorship and Mr. PAUL HEMKE, of Northwestern University, has been appointed instructor.

DR. J. V. DEPORTE has been appointed assistant professor of mathematics at the New York State College for Teachers, Albany.

ASSISTANT professor H. H. CONWELL, of the University of Idaho, has been promoted to an associate professorship of mathematics.

DR. R. L. CHARLES has been promoted to an associate professorship of mathematics at Lehigh University.

MISS M. E. DANIELLS has been promoted to an assistant professorship of mathematics at Iowa State College.

DR. C. M. HEBBERT has been appointed instructor in mathematics at the University of Illinois.

MR. F. W. PARSONS, of Ohio Northern University, has been appointed assistant professor of mathematics and engineering in Whittier College.

DR. H. B. CURTIS and Mr. F. L. BROWN have been appointed instructors in mathematics at Northwestern University. Mr. F. L. KERR has resigned his instructorship to become registrar of the university.

MR. FREDERICK WOOD has been appointed instructor in mathematics at the University of Wisconsin.

AT the College of the City of New York, Mr. HARRY LANGMAN has been appointed instructor in statistics and Dr. H. F. MACNEISH instructor in mathematics.

DR. J. W. CAMPBELL, of Wesley College, Winnipeg, has been appointed associate in mathematics and astronomy at the State University of Iowa.

DR. R. A. ARMS, of Juniata College, has been appointed instructor in mathematics at the University of Pennsylvania.

MR. J. J. TANZOLA has been appointed instructor in mathematics at Cooper Union, New York City.

THE death is reported of Professor A. GOB, of the Athénée royal at Liège, at the age of fifty-one years.

RODOLPHE GUIMARAES, author of Les Mathématiques en Portugal, died in 1918 at the age of fifty-two years.

THE death is reported of Professor G. DEMARTRES, of the University of Lille, at the age of seventy-one years.

DR. O. DANZER, of the Vienna technical school, died March 26, 1919.

PROFESSOR E. BÖTTCHER, of the University of Leipzig, died August 5, 1919, at the age of seventy-two years.

PROFESSOR O. DZIOBEK, of the Charlottenburg technical school, died recently at the age of sixty-three years.

PROFESSOR F. GRAEFE, of the Charlottenburg technical school, died December 2, 1918, at the age of sixty-three years.

PROFESSOR E. NETTO, of the University of Giessen, died recently at the age of seventy-two years.

DR. K. T. REYE, formerly professor at the University of Strassburg, died recently at the age of eighty-one years.

PROFESSOR R. STURM, of the University of Breslau, died recently at the age of seventy-seven years.

DR. J. WELLSTEIN, formerly professor at the University of Strassburg, died June 24, 1919, in his fiftieth year.

MR. P. E. B. JOURDAIN died October 1, 1919, at the age of thirty-eight years.

PROFESSOR JAMES MACLAY, of Columbia University, died November 28, 1919, at the age of fifty-five years.

BOOK CATALOGUES: Galloway and Porter, Cambridge, catalogue 97, list of 495 titles in mathematics and physics.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ADAMS (O. S.). General theory of polyconic projections. (U. S. Coast and Geodetic Survey, Special Publication No. 57.) Washington, Government Printing Office, 1919. 174 pp. Paper. $0.25

BALLANTYNE (J.). The integral calculus. On the integration of the powers of transcendental functions, new methods and theorems, calculation of the Bernoullian numbers, rectification of the logarithmic curve, integration of logarithmic binomials, etc. Boston, J. Ballantyne, 1919. 41 pp. $1.00

FRICKE (R.). Hauptsätze der Differential- und Integralrechnung. 6te Auflage. 1917. M. 7.60

LAMB (H.). An elementary course of infinitesimal calculus. 3d edition revised. Cambridge, University Press, 1919. Demy 8vo. 14 + 530 pp. 20s.

LÆSSER (O.). See SCHWAB (K.).

MANGOLDT (H. VON). Einführung in die höhere Mathematik. 1ter Band. 2te Auflage. 1919. M. 24. 50

MILNE (W. P.). Higher algebra. London, Arnold, 1919. 12 + 586 pp. 8s. 6d.

NICOLETTI (O.). Lezioni di algebra complementare, redatte per uso degli studenti. Pisa, Bemporad, 1919. 8vo. 6 + 661 pp.

REINHARDT (K.). Ueber die Zerlegung der Ebene in Polygone. (Diss.) Frankfurt a. M., 1918.

SAVINO (L.) Una corrispondenza birazionale involutoria fra le rette dello spazio determinata da un fascio-schiera di quadriche. Pavia, tip. succ. Bizzoni, 1919. 8vo. 26 pp.

SCHNEIDER (A.). See Schwab (K.).

SCHWAB (K.) und LESSER (O.). Mathematisches Unterrichtswerk. Geometrie für Lehrer- und Lehrerinnenbildungsanstalten. 1ter Teil, von A. Schneider: Planimetrie, Grundlehren der Kegelschnitte und der analytischen Geometrie. Leipzig, Freytag, 1918. M. 3.40

SUDRE DES CARLIÈRES (R.). L'espace inconditionné. Toulouse, Ed. Privat, 1919. 8vo. 54 pp.

WHITEHEAD (A. N.). An enquiry concerning the principles of natural knowledge. Cambridge, University Press, 1919. 12 + 200 pp. 12s. 6d.

II. ELEMENTARY MATHEMATICS.

AHRENS (W.). Mathematische Spiele. 4te Auflage. Leipzig, Teubner, 1919. Geb. M. 1. 90

AMMERMAN (C.). See FORD (W. B.).

BAFFI (C.). Elementi di algebra, ad uso della terza classe delle scuole tecniche. (Biblioteca di Scienze fisiche, matematiche e naturali.) Torino, Paravia, 1919. 16mo. 104 pp. L. 3.00

DONADT (A.). See LÜBSEN (H. B.).

FORD (W. B.) and AMMERMAN (C.). Teachers' manual, first course in algebra. New York, Macmillan, 1919. 12mo. 314 pp. $2.00

HAWKES (H. E.), LUBY (W. A.) and TOUTON (F. C.). Complete school algebra. Revised edition. Boston, Ginn, 1919. 12mo. 9 + 507 pp. $1.40

HJELMSLEV (J.). Stereometriske Konstruktioner. 2. Oplag. København, Gjellerup, 1919.

KILLION (R. M.). Supplemental problems in arithmetic for rural schools. Third printing, revised. (School Publication No. 4.) Los Angeles, 1919. Paper. 54 pp.

LÜBSEN (H. B.). Ausführliches Lehrbuch der Arithmetik und Algebra zum Selbstunterricht und mit Rücksicht auf die Zwecke des praktischen Lebens. Neubearbeitet von A. Donadt. 28te Auflage. 1917. M. 5.75

LUBY (W. A.). See HAWKES (H. E.).

MANNHEIMER (N.). See REINHARDT (W.).

MARTINI (Z. A.). Ripetitorio di geometria elementare, per le scuole secondarie inferiori. (Biblioteca degli Studenti.) Livorno, R. Giusti, 1919. 8 + 163 pp. L. 2.70

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